Int. J. Heat Mass Transfer. Vol. 12, pp. 173–186. Pergamon Press 1969. Printed in Great Britain

HEAT AND MASS TRANSFER IN A LAMINAR BOUNDARY LAYER ON A FLAT PLATE WITH VARIABLE SUCTION OR INJECTION VELOCITY AND CONSTANT WALL TEMPERATURE

J. L. BANSAL†

Max-Planck-Institut für Strömungsforschung, Göttingen

(*Received* 1 *June* 1968)

Abstract—The velocity and thermal boundary layers over a porous flat plate with variable suction or injection velocity ($\sim x^{-\frac{1}{2}}$) and constant wall temperature, without restricting the scope of the Prandtl number, have been studied. In the first place, a sixth degree velocity profile has been used and the solution is obtained by the von Kármán integral equation. A comparison between the calculated results and the exact solution of Schlichting and Bussmann has been made. Keeping the results of the comparison and the Reynold's analogy in view a sixth degree temperature profile is then considered and the solution is obtained by the heat-flux-equation. The cases of zero and asymptotic suction for velocity and thermal boundary layers are found as the limiting cases of the present study.

NOMENCLATURE

a,	thermal diffusivity, $= k/\rho C_p$;
a_i,	coefficient of η^i in velocity profiles;
b_i,	coefficient of η_t^i in temperature profiles;
C,	defined by equation (3.15);
C_f,	coefficient of skin friction;
C_p,	specific heat at constant pressure;
$D(M)$,	defined by equation (3.7);
$f(M, \eta)$,	defined by equation (3.8);
$H(\Delta)$,	defined by equations (5.17) and (5.18);
$H_t(\Delta, M, M_t)$,	defined by equation (5.10);
k,	coefficient of heat conductivity;
l,	length of the plate:
$I_t(M_t, \eta_t)$,	defined by equation (5.8);
M,	suction parameter, $= \frac{v_0(x)\delta}{\nu}$;
M_t,	defined by equation (5.7);
$Nu(x)$,	local Nusselt number;
$p(M, \Delta)$,	defined by equation (6.9);
Pr,	Prandtl number, $= \mu C_p/k$;
$q(M, \Delta)$,	defined by equation (6.8);
$q_i(M, \Delta)$	Coefficient of M_t^i in equation (6.1) and are defined by equations (6.2) to (6.4);
$Q(x)$,	local heat flux;
$r(M, \Delta)$,	defined by equation (6.18);
$r_i(M, \Delta)$,	coefficient of M_t^i in equation (6.12); and are defined by equations (6.13) to (6.15);
Re_x,	Reynolds number $U_\infty x/\nu$;
$s(M, \Delta)$,	defined by equation (6.19):
t,	dimensionless temperature, $= \frac{T - T_\infty}{T_w - T_\infty}$;
T,	temperature;
u, v,	velocity components in x and y directions respectively;

† Present address: Department of Mathematics, University of Rajasthan, JAIPUR (INDIA).

U_∞,	free stream velocity in x direction;
$v_0(x)$,	prescribed normal velocity at the wall, $v_0(x) > 0$ injection, $v_0(x) < 0$ suction;
x, y,	coordinates along and normal to the wall respectively;
Z_1,	defined by equation (6.7);
Z_2,	defined by equation (6.17).

Greek symbols

$\alpha(x)$,	heat-transfer coefficient, $= \dfrac{Q(x)}{T_w - T_\infty}$;
δ,	velocity boundary-layer thickness;
δ_t,	thermal boundary-layer thickness;
δ^*,	displacement thickness;
Δ,	$= \delta_t/\delta$;
θ,	momentum thickness;
η,	$= y/\delta$;
η_t,	$= y/\delta_t$;
μ,	dynamic viscosity;
ν,	kinematic viscosity, $= \mu/\rho$;
ρ,	density;
τ_w,	shearing stress on the wall:
$\Phi(M)$,	defined by equation (6.5);
$\Phi i(M)$,	defined by equation (5.13) ($i = 0, \ldots, 6$);
$\chi(M)$,	$= \theta/\delta$ [see equation (3.10)];
$\chi(0)$,	$= (\theta/\delta)_{M=0}$.

Subscripts

$0, w$,	conditions at the wall ($y = 0$);
∞,	conditions at outer edge of boundary layer;
I,	injection:
S,	suction.

1. INTRODUCTION

THE EXACT solution for the velocity boundary layer on a flat plate for an incompressible fluid was given by Blasius [1] and the corresponding exact solution for the thermal boundary layer and heat transfer for isothermal and adiabatic walls has been given by Polhausen [2]. The approximate solution for the same problem for thermal boundary layer has been studied by a number of authors such as Kroujiline [3], Eckert [4], Squire [5] and Dienemann [6].

A survey of literature on the laminar boundary layer for exact and approximate solutions has been made by Wuest [7] and Head [8]. The exact solution of the velocity boundary layer over a flat plate with variable suction or injection velocity ($\sim x^{-\frac{1}{2}}$) was first obtained by Schlichting and Bussmann [9] and (independently) by Thwaites [10] and Emmons and Leigh [11]. The corresponding approximate solution for the case of suction has been recently investigated by Morduchow and Reyle [12]. They have found that a sixth degree velocity profile gives a good agreement with the calculated results of Thwaites [10] and Emmons and Leigh [11]. Number of attempts have been made, a detailed bibliography is given by Schlichting [13], by various workers such as Yuan [14], Morduchow [15], Hartnett and Eckert [16] and Koh and Hartnett [17] for the study of heat transfer for compressible and incompressible fluids for isothermal and adiabatic walls with constant or variable suction or injection velocity. In all these studies, for the sake of simplification, either the Prandtl number has been taken as unity, or the calculations are performed with Prandtl number 0·72 the Prandtl number for air.

In the present paper the problem of velocity and thermal boundary layers over a flat plate with variable suction or injection velocity ($\sim x^{-\frac{1}{2}}$) and constant wall temperature has been studied, without restricting the scope of the Prandtl number. A sixth degree velocity profile has been used and first the expressions for velocity profile and the characteristics boundary layer parameters have been derived by the use of the von Kármán integral equation. It is being found that the method gives a good agreement for the case of suction ($M < 0$) Blasius profile ($M = 0$) and for injection ($0 < M \leqslant 3$) with the exact solution of Schlichting and Bussmann, but the results differ widely for $M > 3$, where $M(= [v_0(x)\,\delta(x)]/\nu)$ is the suction parameter. Keeping this thing and Reynolds analogy in view a sixth degree temperature profile is then considered with appropriate boundary conditions and the heat flux equation is employed to determine the relation between Prandtl number Pr, the ratio of the thickness of the temperature and velocity boundary layers $\Delta(= \delta_t(x)/\delta(x))$ and the suction parameter M. The case of zero and asymptotic suction are found as the limiting cases of the present study. The case of zero suction gives a good agreement with the exact solution of Pohlhausen and the case of asymptotic suction gives an interesting relation between Pr and Δ, i.e. $\Delta Pr = 1$.

The assumptions made in the present study are: (a) The potential flow velocity U_∞ is small as regard the velocity of sound, so that the flow is incompressible (Mach number $Ma \to 0$), (b) the coefficient of viscosity μ and the coefficient of conductivity k are small for the validity of the boundary layer assumptions, (c) the temperature difference $T_w - T_\infty$ between the wall and the potential flow is small as regard T_∞, so that the values of μ and k may be taken as constant (i.e. independent of temperature field). Also the Prandtl number $Pr = \mu C_p/k$ is constant, (d) frictional heat and the effect of compressibility are negligible.

2. BASIC EQUATIONS

We consider here the case of a uniform stream U_∞(const) passing over a flat plate with a variable suction or injection velocity $v_0(x)$ normal to the surface of the plate. We shall assume that the x-axis is placed in the plane of the plate in the direction of the flow, the y-axis at right angle to it and to the flow, with the origin at the leading edge. The plate is maintained at a constant temperature T_w and T_∞ being the constant temperature of the undisturbed external flow. The boundary layer equations for the above case are:

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2.2}$$

energy:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2}, \tag{2.3}$$

where

$$a = \frac{\kappa}{\rho C_p} \quad \text{(thermal diffusivity)}.$$

Since the Mach number has been assumed to be small (i.e. incompressible flow) the dissipation term is omitted from the energy equation.

The boundary conditions are

$$\begin{aligned} y = 0:&\quad u = 0, \quad v = v_0(x), \quad T = T_w \\ y = \infty:&\quad u = U_\infty, \quad\quad\quad\quad\quad T = T_\infty, \end{aligned} \tag{2.4}$$

$v_0(x) > 0$ injection; $v_0(x) < 0$ suction.

The velocity field is independent of the temperature field so that the two flow equations (2.1, 2.2) can be solved first and the result can be employed to evaluate the temperature field.

3. APPROXIMATE SOLUTION FOR THE VELOCITY BOUNDARY LAYER

Integrating equation (2.2) between the limits $y = 0$ and $y = \delta$ and making use of (2.1), the well known momentum integral equation for suction or injection in this case is [13]

$$\frac{\tau_w}{\rho} = U_\infty^2\frac{d\theta}{dx} - v_0(x)\,U_\infty, \tag{3.1}$$

where

(shearing stress on the wall)

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \tag{3.2}$$

and

(momentum thickness)

$$\theta = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) \frac{u}{U_\infty}\,\mathrm{d}y. \qquad (3.3)$$

We investigate the problem by Pohlhausen's sixth degree velocity profile with the following boundary conditions

$$\left.\begin{aligned} &\text{At } y = 0: \quad u = 0, \\ &\left(\nu \frac{\partial^2 u}{\partial y^2}\right)_0 = v_0\left(\frac{\partial u}{\partial y}\right)_0, \\ &\left(\nu \frac{\partial^3 u}{\partial y^3}\right)_0 = v_0\left(\frac{\partial^2 u}{\partial y^2}\right)_0, \\ &\text{At } y = \delta: \quad u = U_\infty, \\ &\left(\frac{\partial u}{\partial y}\right)_\delta = \left(\frac{\partial^2 u}{\partial y^2}\right)_\delta = \left(\frac{\partial^3 u}{\partial y^3}\right)_\delta = 0. \end{aligned}\right\} \qquad (3.4)$$

Let

$$\frac{u}{U_\infty} = \sum_{i=0}^{6} a_i \eta^i, \quad 0 \leqslant \eta \leqslant 1, \qquad (3.5)$$

for $\eta > 1, \quad u = U_\infty$,

where

$$\eta = \frac{y}{\delta}.$$

The coefficients in the right-hand side of the expression (3.5) with the help of the boundary conditions (3.4) are obtained as [18],

$$a_0(M) = 0,$$

$$a_1(M) = \frac{120}{D(M)},$$

$$a_2(M) = \frac{60M}{D(M)},$$

$$a_3(M) = \frac{20M^2}{D(M)},$$

$$a_4(M) = \frac{-15(20 + 12M + 3M^2)}{D(M)},$$

$$a_5(M) = \frac{12(30 + 16M + 3M^2)}{D(M)},$$

$$a_6(M) = \frac{-10(12 + 6M + M^2)}{D(M)},$$

where

$$M = \frac{v_0(x)\,\delta(x)}{\nu} \quad \text{(suction parameter)}, \qquad (3.6)$$

and

$$D(M) = 60 + 12M + M^2. \qquad (3.7)$$

Hence

$$\left.\begin{aligned} &\frac{u}{U_\infty} = f(M, \eta) = \sum_{i=0}^{6} a_i(M)\eta^i; \quad 0 \leqslant \eta \leqslant 1 \\ &\text{and} \\ &u = U_\infty; \quad \eta > 1 \end{aligned}\right\} \qquad (3.8)$$

From (3.2) and (3.8), we have

$$\tau_w = \frac{\mu U_\infty}{\delta} a_1(M), \qquad (3.9)$$

and from (3.3) and (3.8)

$$\frac{\theta}{\delta} = \chi(M) = \frac{10}{1001\, D^2(M)}[39400 + 16520M + 2966M^2 + 250M^3 + 10M^4] \qquad (3.10)$$

Let M = const. (i.e. $v_0(x)$ is so adjusted so as to give constant M).

Substituting (3.9) and (3.10) in (3.1), we get

$$\delta \frac{\mathrm{d}\delta}{\mathrm{d}x} = \frac{(a_1(M) + M)\nu}{U_\infty\, \chi(M)} \qquad (3.11)$$

Integrating (3.11) with the condition that at $x = 0, \delta = 0$, we get

$$\delta^2 = \frac{2\nu(a_1(M) + M)}{U_\infty\, \chi(M)}\, x, \qquad (3.12)$$

or

$$v_0^2(x) = \frac{M^2\, \chi(M)}{2(a_1(M) + M)} \frac{\nu U_\infty}{x}. \qquad (3.13)$$

Since M is constant, this shows that $v_0(x) \sim x^{-\frac{1}{2}}$.

Schlichting and Bussmann (9) have taken

$$v_0(x) = -\frac{C}{2}\sqrt{\left(\frac{\nu U_\infty}{x}\right)}, \qquad (3.14)$$

therefore, comparing (3.13) and (3.14), we have

$$C = -M\sqrt{\left[\frac{2\chi(M)}{a_1(M)+M}\right]} \qquad (3.15)$$

when M is positive, $C < 0$ (Injection)
and
when M is negative, $C > 0$ (Suction).

4. THE CHARACTERISTIC BOUNDARY-LAYER PARAMETERS

Displacement thickness

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy = \delta \int_0^1 (1 - f(M,\eta))\, d\eta,$$

therefore

(dimensionless displacement thickness)

$$\delta^*\sqrt{\left(\frac{U_\infty}{\nu x}\right)} = \sqrt{\left[\frac{2(a_1(M)+M)}{\chi(M)}\right]} \times \frac{(120 + 32M + 3M^2)}{7(60 + 12M + M^2)}. \qquad (4.1)$$

From (3.10)

(dimensionless momentum thickness)

$$\theta\sqrt{\left(\frac{U_\infty}{\nu x}\right)} = [2\chi(M)\{a_1(M) + M\}]^{\frac{1}{2}}, \qquad (4.2)$$

From (4.1) and (4.2)

(shape factor)

$$\frac{\delta^*}{\theta} = \frac{1001}{70}\frac{(120 + 32M + 3M^2)(60 + 12M + M^2)}{(39400 + 16520M + 2966M^2 + 250M^3 + 10M^4)}, \qquad (4.3)$$

from (3.9) and (4.1)

(dimensionless shearing stress on the wall)

$$\frac{\tau_w \delta^*}{\mu U_\infty} = \frac{120}{7}\frac{(120 + 32M + 3M^2)}{(60 + 12M + M^2)^2}, \qquad (4.4)$$

and finally for the coefficient of skin friction C_f, we find that

$$C_f\sqrt{\left(\frac{U_\infty l}{\nu}\right)} = 2a_1(M)\sqrt{\left[\frac{2\chi(M)}{a_1(M)+M}\right]}. \qquad (4.5)$$

5. APPROXIMATE SOLUTION FOR THE THERMAL BOUNDARY LAYER

Let δ_t be the thickness of the thermal boundary layer. Integrating (2.3) between the limits $y = 0$ and $y = \delta_t$, we get [19, p. 268]

$$\frac{d}{dx}\int_0^{\delta_t} \frac{u}{U_\infty}\left(\frac{T - T_\infty}{T_w - T_\infty}\right) dy - \frac{v_0(x)}{U_\infty} = -\frac{a}{U_\infty(T_w - T_\infty)}\left(\frac{\partial T}{\partial y}\right)_{y=0}. \qquad (5.1)$$

Introducing the dimensionless temperature t,

$$t = \frac{T - T_\infty}{T_w - T_\infty}$$

equation (5.1) reduces to

$$\frac{d}{dx}\int_0^{\delta_t}\left(t\frac{u}{U_\infty}\right) dy - \frac{v_0(x)}{U_\infty} = -\frac{a}{U_\infty}\left(\frac{\partial t}{\partial y}\right)_{y=0}. \qquad (5.2)$$

For the temperature distribution we consider the following polynomial in $\eta_t = (y/\delta_t)$:

$$t = 1 - \sum_{i=0}^{6} b_i \eta_t^i, \quad 0 \leqslant \eta_t \leqslant 1 \qquad (5.3)$$

where the coefficients b_0 to b_6 are given by the boundary conditions

$\eta_t = 0$: $t = 1$,

$$\left(a\frac{\partial^2 t}{\partial \eta_t^2}\right)_0 = v_0\,\delta_t\left(\frac{\partial t}{\partial \eta_t}\right)_0$$

$$\left(a\frac{\partial^3 t}{\partial \eta_t^3}\right)_0 = v_0\,\delta_t\left(\frac{\partial^2 t}{\partial \eta_t^2}\right)_0$$

and

$\eta_t = 1: \quad t = 0,$

$$\left(\frac{\partial t}{\partial \eta_t}\right)_1 = \left(\frac{\partial^2 t}{\partial \eta_t^2}\right)_1 = \left(\frac{\partial^3 t}{\partial \eta_t^3}\right)_1 = 0. \qquad (5.4)$$

The form of the temperature distribution is so selected as to ensure identical velocity and temperature distribution for $\delta_t = \delta$, as required by the Reynolds analogy for a flat plate at $Pr = 1$.

Let

$$\Delta(x) = \frac{\delta_t(x)}{\delta(x)}, \qquad (5.5)$$

the ratio of the thickness of the temperature and velocity boundary layers.

The coefficients b_0 to b_6, obtained by the boundary conditions (5.4), are given by

$$b_i = a_i(M_t). \qquad (5.6)$$

where

$$M_t = \frac{v_0(x)\,\delta_t(x)}{a} = M\Delta Pr. \qquad (5.7)$$

Hence

$$t = L(M_t, \eta_t) = 1 - \sum_{i=0}^{6} a_i(M_t)\,\eta_t^i. \qquad (5.8)$$

Substituting (3.8) and (5.8) in (5.2), we get

$$\frac{d}{dx}[\delta_t H_t(\Delta, M, M_t)] - \frac{v_0(x)}{U_\infty} = \frac{a\,a_1(M_t)}{U_\infty \delta_t} \qquad (5.9)$$

where

$$H_t(\Delta, M, M_t) = \int_0^1 f(M, \eta)\, L(M_t, \eta_t)\, d\eta_t.\dagger \qquad (5.10)$$

On integrating, we get

$\Delta \leqslant 1: \quad H_t(\Delta, M, M_t)$

$$= \sum_{i=1}^{6} a_i(M)\, \Phi_i(M_t)\, \Delta^i, \qquad (5.11)$$

and

$$\Delta \geqslant 1: \quad H_t(\Delta, M, M_t) = \Phi_0(M_t) - \frac{1}{\Delta}\Phi_0(M) + \sum_{i=1}^{6} a_i(M_t)\, \Phi_i(M)\, \Delta^{-1-i} \qquad (5.12)$$

where

$$\left.\begin{aligned}
\Phi_0(M) &= \frac{1}{7D(M)}(120 + 32M + 3M^2),\\
\Phi_1(M) &= \frac{1}{28D(M)}(100 + 30M + 3M^2),\\
\Phi_2(M) &= \frac{1}{126D(M)}(150 + 48M + 5M^2),\\
\Phi_3(M) &= \frac{1}{168D(M)}(84 + 28M + 3M^2),\\
\Phi_4(M) &= \frac{1}{2310D(M)}(560 + 192M + 21M^2),\\
\Phi_5(M) &= \frac{1}{1386D(M)}(180 + 63M + 7M^2),\\
\Phi_6(M) &= \frac{1}{3003D(M)}(225 + 80M + 9M^2).
\end{aligned}\right\} \qquad (5.13)$$

Now, integrating equation (5.9), we have

$$\delta_t^2 H_t^2(\Delta, M, M_t) = \int_0^{x} \frac{2a}{U_\infty}[a_1(M_t) + M_t] H_t(\Delta, M, M_t) dx. \qquad (5.14)$$

Since $H_t(\Delta, M, M_t)$ is a known function, the preceding equation can be used to determine $\Delta(x)$. The calculation is best performed by successive approximations. Since $\Delta(x)$ is a slowly varying function, therefore to a first approximation we take $\Delta = \text{const}$.

Since M and Δ are constants, therefore for a given fluid (i.e. for a given Prandtl number) M_t is also constant.

Hence, equation (5.14) with the help of (3.12) gives

† H_t should be a function of Δ, M and M_t as

$$\eta = \frac{y}{\delta} = \frac{y}{\delta_t}\frac{\delta_t}{\delta} = \eta_t \Delta.$$

$$\Delta H_t(\Delta, M, M_t) = \frac{M\chi(M)}{M_t} \frac{a_1(M_t) + M_t}{a_1(M) + M}. \quad (5.15)$$

Equation (5.15) shows that Δ is a function of M and Pr and is therefore independent of x. Thus the first approximation is the exact solution of the equation (5.14), and no further approximation is required.

Limiting cases

Case I. When $M \to 0$ (i.e. in the absence of suction or injection), we have from (5.15)

$$\Delta^2 H_t(\Delta, 0, 0) = \frac{\chi(0)}{Pr},$$

or (5.16)

$$\Delta^2 H(\Delta) = \frac{985}{9009} \frac{1}{Pr}.$$

where

$$\Delta \leqslant 1 : H(\Delta) \frac{5}{42}\Delta - \frac{2}{99}\Delta^4 + \frac{1}{77}\Delta^5 - \frac{5}{2002}\Delta^6, \quad (5.17)$$

and

$$\Delta \geqslant 1 : H(\Delta) = \frac{2}{7} - \frac{2}{7}\frac{1}{\Delta} + \frac{5}{42}\frac{1}{\Delta^2} - \frac{2}{99}\frac{1}{\Delta^5} + \frac{1}{77}\frac{1}{\Delta^6} - \frac{5}{2002}\frac{1}{\Delta^7}. \quad (5.18)$$

(i) For moderate values of the Prandtl number the expression

$$\Delta = Pr^{-\frac{1}{3}} \quad (5.19)$$

constitute a good approximation to the solution of equation (5.16).

(ii) For very small Prandtl numbers (i.e. for very large values of Δ), substituting (5.18) in (5.16) and retaining only first term, we get

$$\Delta = 0{\cdot}618\, Pr^{-\frac{1}{2}} \qquad (Pr \to 0). \quad (5.20)$$

(iii) For very large Prandtl numbers (i.e. for very small values of Δ), substituting (5.17) in (5.16) and retaining only first term, we get

$$\Delta = 0{\cdot}972\, Pr^{-\frac{1}{3}} \qquad (Pr \to \infty). \quad (5.21)$$

Case II. When $a_1(M) + M \to 0$ (i.e. the case of asymptotic suction profile), we get

$$M \to -4{\cdot}64437.$$

This value of M is the same as in Case of homogeneous suction [18] and u/U_∞ will be exactly the same function of $[\rho y_0(x) y/\nu]$ as in Case of homogeneous suction which is already pointed out by Morduchow and Reyle [12] and is in exact agreement with the implications of the asymptotic suction profile [9–11].

As Δ and $\chi(M)$ are finite, relation (5.15) imply that in this case $a_1(M_t) + M_t$ should also tend to zero, this gives

$$\Delta = \frac{1}{Pr}. \quad (5.22)$$

Equation (5.22) can also be obtained, independently from the basic equations (2.2) and (2.3) by taking u and T, for the asymptotic suction profile, independent of x.

6. SOLUTIONS OF EQUATION (5.15) FOR DIFFERENT VALUES OF THE PRANDTL NUMBER

When $\Delta \leqslant 1$, substituting the values of $H_t(\Delta, M, M_t)$ $\chi(M)$, $a_1(M_t)$ and $a_1(M)$ from (5.11), (3.10) and (3.6) respectively in (5.15) and after simplification, we get

$$\sum_{i=1}^{3} q_i(M, \Delta) M_t^i = -120, \quad (6.1)$$

$$q_1(M, \Delta) = 60 - \Phi(M)\left[\frac{25}{7}a_1(M)\Delta^2 + \frac{25}{21}a_2(M)\Delta^3 + \frac{1}{2}a_3(M)\Delta^4 + \frac{8}{33}a_4(M)\Delta^5 + \frac{10}{77}a_5(M)\Delta^6 + \frac{75}{1001}a_6(M)\Delta^7\right], \quad (6.2)$$

$$q_2(M,\Delta) = 12 - \Phi(M)\left[\frac{15}{14}a_1(M)\Delta^2 + \frac{8}{21}a_2(M)\Delta^3 + \frac{1}{6}a_3(M)\Delta^4 + \frac{32}{385}a_4(M)\Delta^5 + \frac{1}{22}a_5(M)\Delta^6 + \frac{80}{3003}a_6(M)\Delta^7\right], \quad (6.3)$$

$$q_3(M,\Delta) = 1 - \Phi(M)\left[\frac{3}{28}a_1(M)\Delta^2 + \frac{5}{126}a_2(M)\Delta^3 + \frac{1}{56}a_3(M)\Delta^4 + \frac{1}{110}a_4(M)\Delta^5 + \frac{1}{198}a_5(M)\Delta^6 + \frac{3}{1001}a_6(M)\Delta^7\right], \quad (6.4)$$

such that

$$\Phi(M) = \frac{a_1(M) + M}{M\chi(M)}. \quad (6.5)$$

Equation (6.1) can be reduced to

$$Z_1^3 + 3p(M,\Delta)Z_1 + 2q(M,\Delta) = 0, \quad (6.6)$$

where

$$Z_1 = M_t + \frac{q_2(M,\Delta)}{3q_3(M,\Delta)}, \quad (6.7)$$

$$2q(M,\Delta) = \frac{2q_2^3}{27q_3^3} - \frac{q_1q_2}{3q_3^2} + \frac{120}{q_3}, \quad (6.8)$$

and

$$3p(M,\Delta) = \frac{3q_1q_3 - q_2^2}{3q_3^2}. \quad (6.9)$$

The values of $q^2 + p^3$ are tabulated for different values of $\Delta \leqslant 1$ and for positive and negative values of M. It is found that $q^2 + p^3$ is.positive in all such cases. Therefore, there exists only one real value of Z_1 (Cardon's solution) which is given by

$$Z_1 = [-q + (q^2 + p^3)^{\frac{1}{2}}]^{\frac{1}{3}} + [-q - (q^2 + p^3)^{\frac{1}{2}}]^{\frac{1}{3}}. \quad (6.10)$$

Therefore, from (6.7) the Prandtl number can be obtained as a function of M and Δ i.e.

$$\Delta \leqslant 1: \quad Pr = \frac{1}{M\Delta}\left[Z_1 - \frac{q_2(M,\Delta)}{3q_3(M,\Delta)}\right]. \quad (6.11)$$

When $\Delta \geqslant 1$, equation (5.15) reduces to

$$\sum_{i=1}^{3} r_i(M,\Delta)M_t^i = -120, \quad (6.12)$$

where

$$r_1(M,\Delta) = 60 + \Phi(M)\left[60\Phi_0(M) - \frac{120}{7}\Delta - 120\Phi_1(M)\Delta^{-1} + 300\Phi_4(M)\Delta^{-4} - 360\Phi_5(M)\Delta^{-5} + 120\Phi_6(M)\Delta^{-6}\right], \quad (6.13)$$

$$r_2(M,\Delta) = 12 + \Phi(M)\left[12\Phi_0(M) - \frac{32}{7}\Delta - 60\Phi_2(M)\Delta^{-2} + 180\Phi_4(M)\Delta^{-4} - 192\Phi_5(M)\Delta^{-5} + 60\Phi_6(M)\Delta^{-6}\right], \quad (6.14)$$

and

$$r_3(M,\Delta) = 1 + \Phi(M)\left[\Phi_0(M) - \frac{3}{7}\Delta - 20\Phi_3(M)\Delta^{-3} + 45\Phi_4(M)\Delta^{-4} - 36\Phi_5(M)\Delta^{-5} + 10\Phi_6(M)\Delta^{-6}\right]. \quad (6.15)$$

Proceeding as in the case of $\Delta \leqslant 1$, from equation (6.12), we get

$$\Delta \geqslant 1: \quad Pr = \frac{1}{M\Delta}\left[Z_2 - \frac{r_2(M,\Delta)}{3\,r_3(M,\Delta)}\right], \qquad (6.16)$$

where

$$Z_2 = [-r + (r^2 + s^3)^{\frac{1}{2}}]^{\frac{1}{3}} + [-r - (r^2 + s^3)^{\frac{1}{2}}]^{\frac{1}{3}}, \qquad (6.17)$$

$$2r(M,\Delta) = \frac{2\,r_2^3}{27\,r_3^3} - \frac{r_1\,r_2}{3\,r_3^2} + \frac{120}{r_3}, \qquad (6.18)$$

and

$$3\,s(M,\Delta) = \frac{3\,r_1\,r_3 - r_2^2}{3\,r_3^2}. \qquad (6.19)$$

7. HEAT TRANSFER

If $Q(x)$ denotes the quantity of heat exchanged between the plate and the fluid per unit area and time at a point x, then [13, p. 262]

$$Q(x) = \alpha(x) \times (T_w - T_\infty) = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}. \qquad (7.1)$$

Introducing dimensionless quantities, we obtain a local dimensionless coefficient of heat transfer which is known as the Nusselt number as

$$Nu(x) = \frac{\alpha(x)l}{k} = -\left(\frac{\partial t}{\partial y/l}\right)_{y=0}. \qquad (7.2)$$

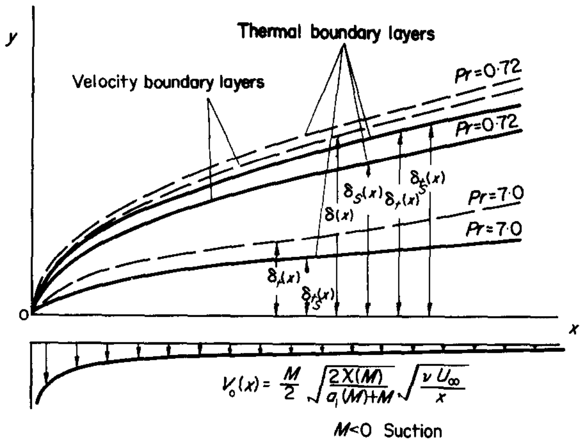

FIG. 1. Velocity and thermal boundary layers on a flat plate with variable suction ($\sim x^{-\frac{1}{2}}$) and constant wall temperature T_w. (Subscript S denotes suction).

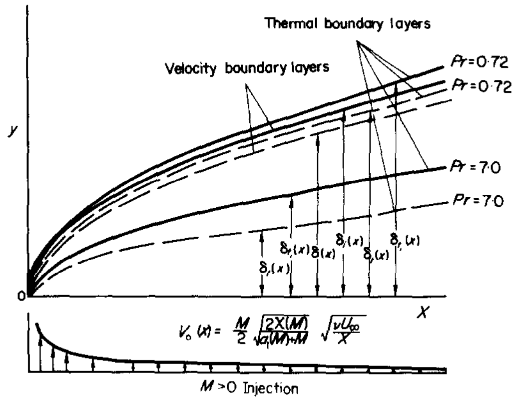

FIG. 2. Velocity and thermal boundary layers on a flat plate with variable injection ($\sim x^{-\frac{1}{2}}$) and constant wall temperature T_w. (Subscript I denotes injection).

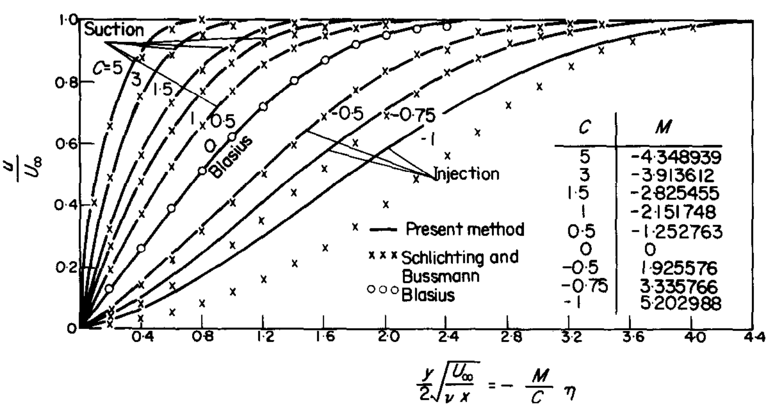

FIG. 3. Velocity profiles for various values of the parameter $C = -M\surd[2\chi(M)/[(a_1(M) + M)]$.

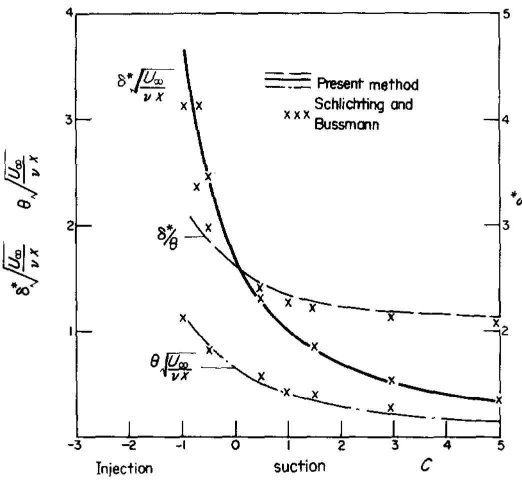

FIG. 4. The dimensionless displacement thickness $\delta^* \sqrt{[U_\infty/\nu x]}$ the momentan thickness $\theta\sqrt{[U_\infty/\nu x]}$ and the shape factor δ^*/θ against the parameter $C = -M\sqrt{[2\chi(M)/(a_1(M) + M)]}$ (M is the suction parameter).

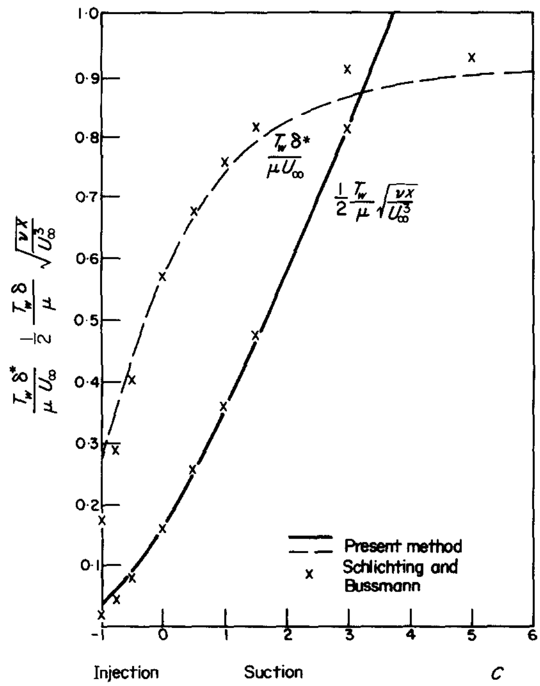

FIG. 5. Dimensionless shearing stress $T_w\delta^*/\mu U_\infty$ or $\frac{1}{2}T_w/\mu\sqrt{[\nu x/U_\infty^3]}$ against the parameter $C = -M\sqrt{[2\chi(M)/(a_1(M) + M)]}$ (M is the suction parameter).

From (5.3) and (7.2), we obtain

$$\frac{Nu(x)}{\sqrt{(Re_x)}} = \frac{a_1(M_t)}{2}\left[\frac{2\chi(M)}{a_1(M) + M}\right]^{\frac{1}{2}}\frac{1}{\Delta}, \quad (7.3)$$

where

$$Re_x = \frac{U_\infty x}{\nu}. \quad \text{(Reynolds number)}$$

when $M \to 0$, (7.3) gives

$$\frac{Nu(x)}{\sqrt{(Re_x)}} = \frac{0{\cdot}330}{\Delta}. \quad (7.4)$$

Relation (7.4), for different ranges of the Prandtl number, with the help of (5.20), (5.19) and (5.21), can be written as

$Nu(x) = 0{\cdot}534\sqrt{(Pr)}\sqrt{(Re_x)}$ $(Pr \to 0)$

$Nu(x) = 0{\cdot}330\ {}^3\sqrt{(Pr)}\sqrt{(Re_x)}$ (for moderate values of the Prandtl number)

$Nu(x) = 0{\cdot}339\ {}^3\sqrt{(Pr)}\sqrt{(Re_x)}$ $(Pr \to \infty)$.

The corresponding values of the Nusselt number from the exact solutions are [13, p. 285]

$Nu(x) = 0{\cdot}564\sqrt{(Pr)}\sqrt{(Re_x)}$ $(Pr \to 0)$

$Nu(x) = 0{\cdot}332\ \sqrt[3]{(Pr)}\sqrt{(Re_x)}$ $(0{\cdot}6 < Pr < 10)$

$Nu(x) = 0{\cdot}339\ \sqrt[3]{(Pr)}\sqrt{(Re_x)}$ $(Pr \to \infty)$.

The method of calculating the thermal boundary layer and in particular the local Nusselt number is now as follows:

(a) For negative and positive values of the suction parameter M and for small and large values of Δ, the relation between Prandtl number and Δ can be obtained from Figs. 6 and 7 which are based on equations (6.11) and (6.16),

(b) For a given Prandtl number and for a given value of the suction parameter M the value of $\Delta = \delta_t(x)/\delta(x)$ can be obtained either from Fig. 6 or from Fig. 7 as the case may be,

(c) For a given value of the suction parameter M, the thickness $\delta(x)$ of the velocity boundary

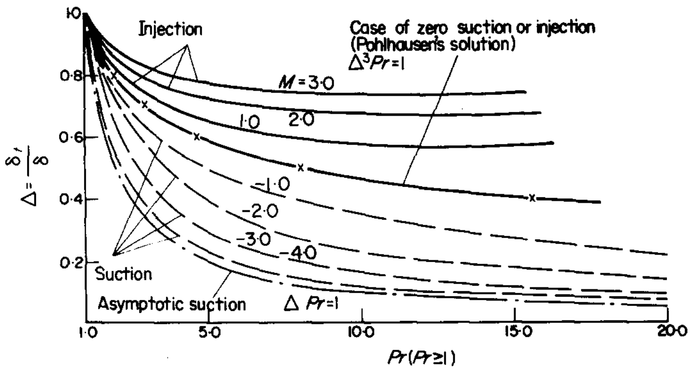

FIG. 6. Prandtl number against the parameter Δ (the ratio of the thickness of the temperature and velocity boundary layers) for various values of M (suction parameter). $M < 0$ suction, $M > 0$ injection.

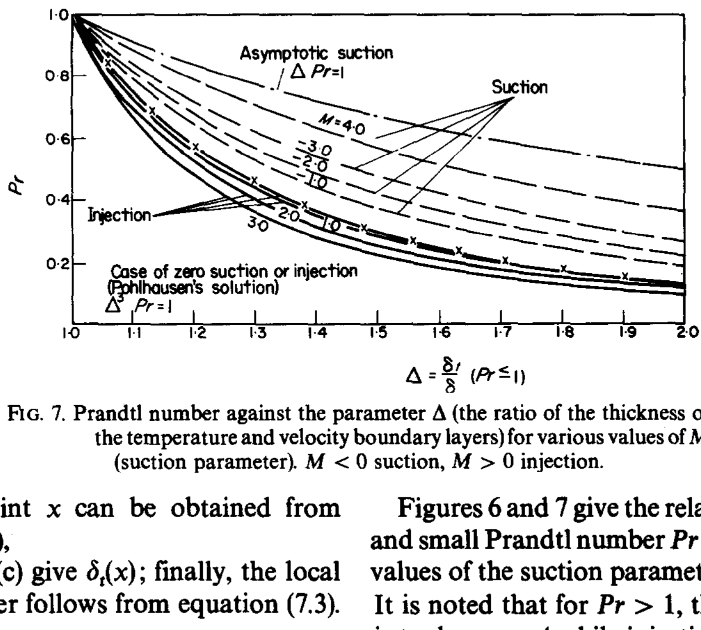

FIG. 7. Prandtl number against the parameter Δ (the ratio of the thickness of the temperature and velocity boundary layers) for various values of M (suction parameter). $M < 0$ suction, $M > 0$ injection.

layer at a point x can be obtained from equation (3.12),

(d) Steps (b) and (c) give $\delta_t(x)$; finally, the local Nusselt number follows from equation (7.3).

8. CONCLUSION

In Figs. 3–5 the velocity profiles and the characteristic boundary layer parameters are compared with those of Schlichting and Bussmann. It is being found that for the case of suction ($M < 0$), Blasius profile ($M = 0$) and for injection ($0 < M \leqslant 3$) the results are in good agreement with the exact solution of Schlichting and Bussmann, but for $M > 3$ the sixth degree polynomial does not give a fair agreement and the results differ widely.

Figures 6 and 7 give the relation between large and small Prandtl number Pr and Δ, for different values of the suction parameter M, respectively. It is noted that for $Pr > 1$, the effect of suction is to decrease Δ while injection increases it. The opposite phenomenon happens for $Pr < 1$. For $Pr = 1$, we have $\Delta = 1$ (Reynolds analogy). The study leads to the fact that suction decreases the velocity as well as thermal boundary layers while injection increases them. For two fluids $Pr = 0{\cdot}72$ (air) and $Pr = 7{\cdot}0$ (water) the thermal and velocity boundary layers for suction ($M = -1$) and injection ($M = 1$) are shown in Figs. 1 and 2 respectively.

In Figs. 8 and 9 the local Nusselt number is plotted against the small and large values of the

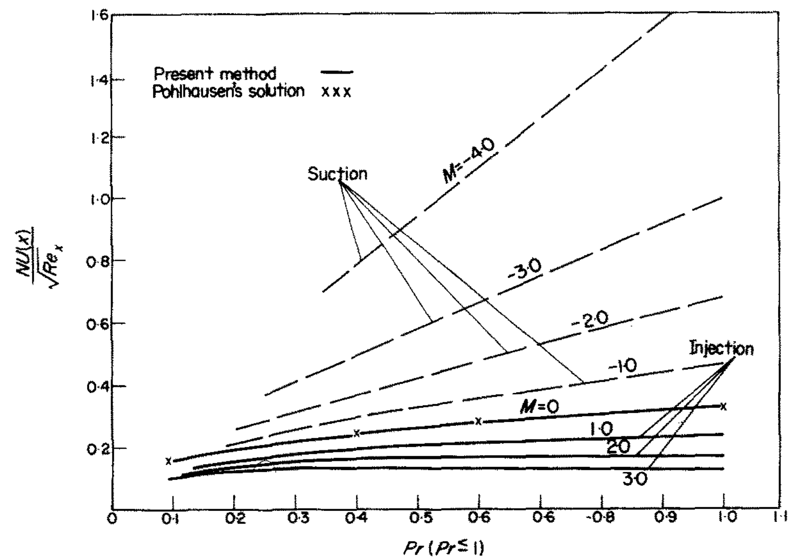

FIG. 8. Local Nusselt number $Nu(x)$ plotted against the Prandtl number Pr for positive and negative values of the suction parameter M.

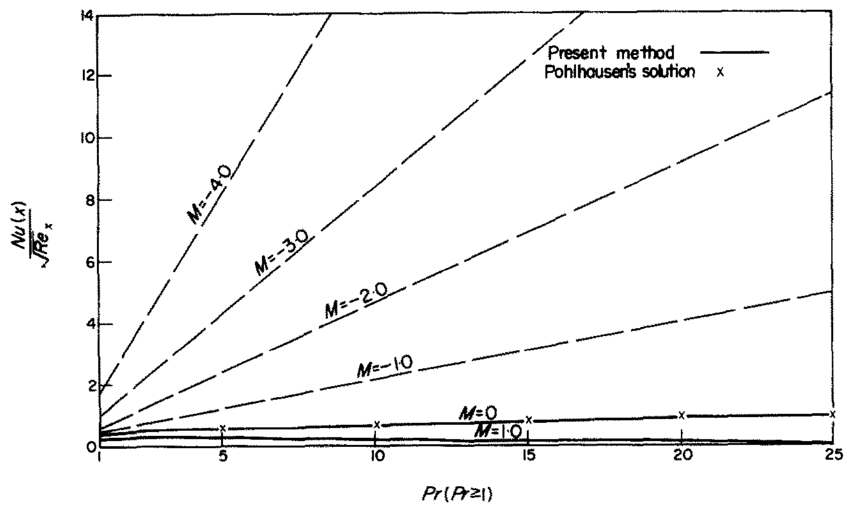

FIG. 9. Local Nusselt number $Nu(x)$ plotted against the Prandtl number Pr for positive and negative values of the suction parameter M.

Prandtl number, for various values of the parameter M. In case of injection, the reduction in Nusselt number shows that a very effective reduction of the wall surface temperature can be obtained by injecting a small amount of fluid. This cooling process is known as transpiration or sweat cooling.

For the case of asymptotic suction we find that $\Delta = 1/Pr$. In the case of impermeable wall the expressions for the Nusselt number for various ranges of the Prandtl number gives a very good agreement with the known exact solutions.

It is expected that the present study will give a fair idea of the nature of the thermal boundary layer and heat transfer on a porous flat plate with suction or injection ($\sim x^{-\frac{1}{2}}$) for a wide range of Prandtl numbers when the fluid is incompressible and the frictional heat is neglected.

ACKNOWLEDGEMENTS

The author is indebted to Dr. W. Wuest, Aerodynamische Versuchsanstalt Göttingen, for his guidance and help during the preparation of this paper.

He also wishes to thank DAAD for awarding a scholarship which enabled him to carry out this work at Max-Planck-Institut für Strömungsforschung, Göttingen.

REFERENCES

1. H. Blasius, Grenzschichten in Flüssigkeiten mit kleiner Reibung, *Z. Math. Phys.* **56**, 1–37 (1908).
2. E. Pohlhausen, Der Wärmeaustausch zwischen festen Körpern und Flüssigkeiten mit kleiner Reibung und kleiner Wärmeleitung. *Z. Angew. Math. Mech.* **1**, 115 (1921).
3. G. Kroüjiline, Investigation de la couche-limite thermiqué, *Techn. Phys. U.S.S.R.* Bd. 3, 183–194 (1936).
4. E. Eckert, Die Berechnung des Wärmeüberganges in der laminaren Grenzschicht umströmter Körper, *ForschHft. Ver. Dt. Ing.* 416 (1942).
5. H. B. Squire, Heat transfer calculation for aerofoils, ARC RM 1986 (1942).
6. W. Dienemann, Berechnung des Wärmeüberganges an laminar umströmten Körpern mit konstanter und ortsveränderlicher Wandtemperatur, *Z. Angew. Math. Mech.* **33**, 89–109 (1953).
7. W. Wuest, Survey of calculation methods of laminar boundary layers with suction in incompressible flow, in *Boundary Layer and Flow Control—Its Principles and Application*, edited by G. V. Lachmann, Vol. 2, pp. 771–800. Pergamon Press, Oxford (1961).
8. M. R. Head, Approximate methods of calculating the two-dimensional laminar boundary layer with suction, in *Boundary Layer and Flow Control—Its Principles and and Applications*, edited by G. V. Lachmann, Vol. 2, pp. 801–841. Pergamon Press, Oxford (1961).
9. H. Schlichting and K. Bussmann, Exakte Lösungen für die Laminare Reibungsschicht mit Absaugung und Ausblasen, *Schr. Dt. Akad. LuftForsch.* **7** 25–69, **B**(2), (1943).
10. B. Thwaites, An exact solution of the boundary layer equations under particular conditions of porous surface suction, Aero, Res. Council, Rept. and Memo. 2241 (1946).
11. H. W. Emmons and D. C. Leigh, Tabulation of the blasius function with blowing and suction, Aero. Res. Council, C.P. No. 157 (1954).
12. M. Morduchow and S. P. Reyle, On the compressible laminar boundary layer with suction, *Int. J. Heat Mass Transfer* **10**, 1233 (1967).
13. H. Schlichting, *Boundary Layer Theory*. McGraw-Hill, New York (1968).
14. S. W. Yuan, Heat transfer in laminar compressible boundary layer on a porous flat plate with fluid in-injection, *Jn. Aeronaut. Sci.* **16**(12), 741–748 (1949).
15. M. Morduchow, On heat transfer over a sweat-cooled surface in laminar compressible flow with a pressure gradient, *Jn. Aeronaut. Sci.* **19**(10), 704–712 (1952).
16. J. P. Hartnett and E. R. G. Eckert, Mass transfer cooling in a laminar boundary layer with constant fluid properties, *Trans. Am. Soc. Mech. Engrs* **79**, 247 (1957).
17. J. C. Y. Koh and J. P. Hartnett, Skin friction and heat transfer for incompressible laminar flow over porous wedges with suction and variable wall temperature, *Int. J. Heat Mass Transfer* **2**, 185 (1961).
18. J. L. Bansal, Boundary layer on a flat plate with homogeneous suction. *Z. Angew. Math. Phys.* in press.
19. E. Eckert, *Wärme- und Stoffaustausch*. Springer, Berlin (1959).

Résumé—Les couches limites de vitesse et de température sur une plaque plane poreuse avec une vitesse d'aspiration ou d'injection variable ($\sim x^{-\frac{1}{2}}$) et une température pariétale constant ont été étudiées sans restreindre la gamme de variation du nombre de Prandtl. Tout d'abord, on a employé un profil de vitesse du sixième degré et la solution a été obtenue par l'équation intégrale de Von Kármán. On a comparé les résultats calculés et la solution exacte de Schlichting et Bussmann. En tenant compte des résultats de la comparaison et de l'analogie de Reynolds, on considère alors un profil de température du sixième degré et l'on obtient la solution par l'équation du flux de chaleur. Les cas de l'aspiration nulle et asymptotique pour des couches limites de vitesse et de température sont obtenus comme cas limites de l'étude actuelle.

Zusammenfassung—Die Geschwindigkeits- und Temperaturgrenzschichten an einter porösen ebenen Platte mit veränderlicher Absauge- oder Ausblasgeschwindigkeit ($\sim x^{-\frac{1}{2}}$) und konstanter Wandtemperatur sind ohne Beschränkung des Bereichs der Prandtlzahl untersucht worden. Ein Geschwindigkeitsprofil sechsten Grades ist zunächst benutzt worden und die Lösung wurde durch die von Kármánsche Integralbedingung gewonnen. Die berechneten Ergebnisse sind mit der exakten Lösung von Schlichting und

Bussmann verglichen worden. Unter Berücksichtigung der Ergebnisse des Vergleichs und der Reynolds-Analogie wurde dann ein Temperaturprofil sechsten Grades betrachtet und eine Lösung mit Hilfe der Wärmeflussgleichung erhalten. Als Grenzfälle der vorliegenden Untersuchung wurden die Fälle verschwindender und homogener Absaugung für die Geschwindigkeits- und Temperaturgrenzschichten gefunden.

Аннотация—Исследовались скоростные и температурные поля в пограничном слое плоской пластины при различной скорости отсоса и вдува ($\sim X^{-\frac{1}{2}}$) через пористую изотермическую стенку без ограничения диапазона числа Прандтля. В основу приближенного решения по интегральному методу Кармана положена аппроксимация профиля скорости параболой шестой степени. Проведено сравнение результатов расчета с точным решением Шлихтинга и Бусмана. Использовано приближение шестой степени для температурного профиля совместно с аналогией Рейнольдса и на этой основе получена формула для тепловых потоков. Случаи нулевой и асимптотической скорости отсоса для тепловых пограничных слоев, рассмотренные в данном исследовании, являются предельными.